Exercises 14.1

Domain, Range, and Level Curves
In Exercises 1–4, find the specific function values.
1. \( f(x, y) = x^2 + xy^3 \)
   a. \( f(0, 0) \)  
   b. \( f(-1, 1) \)  
   c. \( f(2, 3) \)  
   d. \( f(-3, -2) \)
2. \( f(x, y) = \sin(xy) \)
   a. \( f\left(\frac{\pi}{6}, \frac{\pi}{6}\right) \)  
   b. \( f\left(-3, \frac{\pi}{12}\right) \)  
   c. \( f\left(\pi, \frac{1}{4}\right) \)  
   d. \( f\left(-\frac{\pi}{2}, -7\right) \)
3. \( f(x, y, z) = \frac{x - y}{y^2 + z^2} \)
   a. \( f(3, -1, 2) \)  
   b. \( f\left(1, \frac{1}{2}, -\frac{1}{4}\right) \)  
   c. \( f\left(0, \frac{1}{3}, 0\right) \)  
   d. \( f(2, 2, 100) \)
4. \( f(x, y, z) = \sqrt{49 - x^2 - y^2 - z^2} \)
   a. \( f(0, 0, 0) \)  
   b. \( f(2, -3, 6) \)  
   c. \( f(-1, 2, 3) \)  
   d. \( f\left(\frac{4}{\sqrt{2}}, \frac{5}{\sqrt{2}}, \frac{6}{\sqrt{2}}\right) \)

In Exercises 5–12, find and sketch the domain for each function.
5. \( f(x, y) = \sqrt{y - x - 2} \)
6. \( f(x, y) = \ln(x^2 + y^2 - 4) \)
7. \( f(x, y) = \frac{(x - 1)(y + 2)}{(y - x)(y - x^2)} \)
8. \( f(x, y) = \frac{\sin(xy)}{x^2 + y^2 - 25} \)
9. \( f(x, y) = \cos^{-1}(y - x^2) \)
10. \( f(x, y) = \ln(ey + x - y - 1) \)
11. \( f(x, y) = \sqrt{(x^2 - 4)(y^2 - 9)} \)
12. \( f(x, y) = \frac{1}{\ln(4 - x^2 - y^2)} \)

In Exercises 13–16, find and sketch the level curves \( f(x, y) = c \) on the same set of coordinate axes for the given values of \( c \). We refer to these level curves as a contour map.
13. \( f(x, y) = x + y - 1, \quad c = -3, -2, -1, 0, 1, 2, 3 \)
14. \( f(x, y) = x^2 + y^2, \quad c = 0, 1, 4, 9, 16, 25 \)
15. \( f(x, y) = xy, \quad c = -9, -4, -1, 0, 1, 4, 9 \)
16. \( f(x, y) = \sqrt{25 - x^2 - y^2}, \quad c = 0, 1, 2, 3, 4 \)

In Exercises 17–30, (a) find the function’s domain, (b) find the function’s range, (c) describe the function’s level curves, (d) find the boundary of the function’s domain, (e) determine if the domain is an open region, a closed region, or neither, and (f) decide if the domain is bounded or unbounded.
17. \( f(x, y) = y - x \)
18. \( f(x, y) = \sqrt{y - x} \)
19. \( f(x, y) = 4x^2 + 9y^2 \)
20. \( f(x, y) = x^2 - y^2 \)
21. \( f(x, y) = xy \)
22. \( f(x, y) = y/x^2 \)
23. \( f(x, y) = \frac{1}{\sqrt{16 - x^2 - y^2}} \)
24. \( f(x, y) = \sqrt{9 - x^2 - y^2} \)
25. \( f(x, y) = \ln(x^2 + y^2) \)
26. \( f(x, y) = e^{(x^2+y^2)} \)
27. \( f(x, y) = \sin^{-1}(y - x) \)
28. \( f(x, y) = \tan^{-1}\left(\frac{y}{x}\right) \)
29. \( f(x, y) = \ln(x^2 + y^2 - 1) \)
30. \( f(x, y) = \ln(9 - x^2 - y^2) \)

Matching Surfaces with Level Curves
Exercises 31–36 show level curves for the functions graphed in (a)–(f) on the following page. Match each set of curves with the appropriate function.

31. [Image of a level curve graph]
32. [Image of a level curve graph]
33. [Image of a level curve graph]
34. [Image of a level curve graph]
35. [Image of a level curve graph]
36. [Image of a level curve graph]
Functions of Two Variables
Display the values of the functions in Exercises 37–48 in two ways: (a) by sketching the surface \( z = f(x, y) \) and (b) by drawing an assortment of level curves in the function's domain. Label each level curve with its function value.

37. \( f(x, y) = y^2 \)
38. \( f(x, y) = \sqrt{x} \)
39. \( f(x, y) = x^2 + y^2 \)
40. \( f(x, y) = \sqrt{x^2 + y^2} \)
41. \( f(x, y) = x^2 - y \)
42. \( f(x, y) = 4 - x^2 - y^2 \)
43. \( f(x, y) = 4x^2 + y^2 \)
44. \( f(x, y) = 6 - 2x - 3y \)
45. \( f(x, y) = 1 - |y| \)
46. \( f(x, y) = 1 - |x| - |y| \)
47. \( f(x, y) = \sqrt{x^2 + y^2} + 4 \)
48. \( f(x, y) = \sqrt{x^2 + y^2} - 4 \)

Finding Level Curves
In Exercises 49–52, find an equation for and sketch the graph of the level curve of the function \( f(x, y) \) that passes through the given point.

49. \( f(x, y) = 16 - x^2 - y^2 \), \( (2 \sqrt{2}, \sqrt{2}) \)
50. \( f(x, y) = \sqrt{x^2 - 1} \), \( (1, 0) \)
51. \( f(x, y) = \sqrt{x^2 + y^2} - 3 \), \( (3, -1) \)
52. \( f(x, y) = \frac{2y - x}{x + y + 1} \), \( (-1, 1) \)

Sketching Level Surfaces
In Exercises 53–60, sketch a typical level surface for the function.

53. \( f(x, y, z) = x^2 + y^2 + z^2 \)
54. \( f(x, y, z) = \ln(x^2 + y^2 + z^2) \)
55. \( f(x, y, z) = x + z \)
56. \( f(x, y, z) = x \)
57. \( f(x, y, z) = x^2 + y^2 \)
58. \( f(x, y, z) = y^2 + z^2 \)
59. \( f(x, y, z) = z - x^2 - y^2 \)
60. \( f(x, y, z) = (x^2/25) + (y^2/16) + (z^2/9) \)

Finding Level Surfaces
In Exercises 61–64, find an equation for the level surface of the function through the given point.

61. \( f(x, y, z) = \sqrt{x - y - \ln z} \), \( (3, -1, 1) \)
62. \( f(x, y, z) = \ln(x^2 + y^2 + z^2) \), \( (-1, 2, 1) \)

71. \( f(x, y) = \sin(x + 2 \cos y) \), \( -2\pi \leq x \leq 2\pi, -2\pi \leq y \leq 2\pi \)
72. \( f(x, y) = e^{(x-y)} \sin(x^2 + y^2) \), \( 0 \leq x \leq 2\pi, -2\pi \leq y \leq 2\pi \)
In Exercises 65–68, find and sketch the domain of \( f \). Then find an equation for the level curve or surface of the function passing through the given point.

65. \( f(x, y) = \sum_{n=0}^{\infty} \left( \frac{x}{y} \right)^n \), \((1, 2)\)

66. \( g(x, y, z) = \sum_{n=0}^{\infty} \frac{(x + y)^n}{n!z^n} \), \((\ln 4, \ln 9, 2)\)

67. \( f(x, y) = \int_0^y \frac{dt}{\sqrt{1 - t^2}} \), \((0, 1)\)

68. \( g(x, y, z) = \int_0^y \frac{dt}{1 + t^2} + \int_0^x \frac{d\theta}{\sqrt{4 - \theta^2}} \), \((0, 1, \sqrt{3})\)

**COMPUTER EXPLORATIONS**

Use a CAS to perform the following steps for each of the functions in Exercises 69–72.

a. Plot the surface over the given rectangle.

b. Plot several level curves in the rectangle.

c. Plot the level curve of \( f \) through the given point.

69. \( f(x, y) = x \sin \frac{y}{2} + y \sin 2x \), \( 0 \leq x \leq 5\pi, \ 0 \leq y \leq 5\pi \), \( P(3\pi, 3\pi) \)

70. \( f(x, y) = (\sin x)(\cos y)y^{\sqrt{2t+1/2}} \), \( 0 \leq x \leq 5\pi, \ 0 \leq y \leq 5\pi \), \( P(4\pi, 4\pi) \)

### 14.1 FUNCTIONS OF SEVERAL VARIABLES

1. (a) \( f(0, 0) = 0 \)
   
   (d) \( f(-3, -2) = 33 \)

2. (a) \( f(2, \frac{\pi}{6}) = \frac{\sqrt{3}}{2} \)
   
   (d) \( f(-\frac{\pi}{2}, -7) = -1 \)

3. (a) \( f(3, -1, 2) = \frac{4}{3} \)
   
   (d) \( f(2, 2, 100) = 0 \)

4. (a) \( f(0, 0, 0) = 7 \)
   
   (d) \( f\left(\frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3}\right) = \frac{\sqrt{21}}{2} \)

5. Domain: all points \((x, y)\) on or above the line \( y = x + 2 \)

6. Domain: all points \((x, y)\) outside the circle \( x^2 + y^2 = 4 \)

Use a CAS to plot the implicitly defined level surfaces in Exercises 73–76.

73. \( 4 \ln (x^2 + y^2 + z^2) = 1 \)

74. \( x^2 + z^2 = 1 \)

75. \( x + y^2 - 3z^2 = 1 \)

76. \( \sin \left(\frac{x}{2}\right) - (\cos y)\sqrt{x^2 + z^2} = 2 \)

**Parametrized Surfaces** Just as you describe curves in the plane parametrically with a pair of equations \( x = f(t), y = g(t) \) defined on some parameter interval \( I \), you can sometimes describe surfaces in space with a triple of equations \( x = f(u, v), y = g(u, v), z = h(u, v) \) defined on some parameter rectangle \( a \leq u \leq b, c \leq v \leq d. \) Many computer algebra systems permit you to plot such surfaces in parametric mode. (Parametrized surfaces are discussed in detail in Section 16.5.) Use a CAS to plot the surfaces in Exercises 77–80. Also plot several level curves in the \( xy \)-plane.

77. \( x = u \cos v, \ y = u \sin v, \ z = u, \ 0 \leq u \leq 2, \ 0 \leq v \leq 2\pi \)

78. \( x = u \cos v, \ y = u \sin v, \ z = u, \ 0 \leq u \leq 2, \ 0 \leq v \leq 2\pi \)

79. \( x = (2 + \cos u) \cos v, \ y = (2 + \cos u) \sin v, \ z = \sin u, \ 0 \leq u \leq 2\pi, \ 0 \leq v \leq 2\pi \)

80. \( x = 2 \cos u \cos v, \ y = 2 \cos u \sin v, \ z = 2 \sin u, \ 0 \leq u \leq 2\pi, \ 0 \leq v \leq \pi \)
7. Domain: all points \((x, y)\) not lying on the graph of \(y = x\) or \(y = x^3\)

8. Domain: all points \((x, y)\) not lying on the graph of \(x^2 + y^2 = 25\)

9. Domain: all points \((x, y)\) satisfying \(x^2 - 1 \leq y \leq x^2 + 1\)

10. Domain: all points \((x, y)\) satisfying \((x - 1)(y + 1) > 0\)

11. Domain: all points \((x, y)\) satisfying \((x - 2)(x + 2)(y - 3)(y + 3) \geq 0\)

12. Domain: all points \((x, y)\) inside the circle \(x^2 + y^2 = 4\) such that \(x^2 + y^2 \neq 3\)

13. 

14.
17. (a) Domain: all points in the xy-plane  
   (b) Range: all real numbers  
   (c) level curves are straight lines $y - x = c$ parallel to the line $y = x$  
   (d) no boundary points  
   (e) both open and closed  
   (f) unbounded

18. (a) Domain: set of all $(x,y)$ so that $y - x \geq 0 \Rightarrow y \geq x$  
   (b) Range: $z \geq 0$  
   (c) level curves are straight lines of the form $y - x = c$ where $c \geq 0$  
   (d) boundary is $\sqrt{y - x} = 0 \Rightarrow y = x$, a straight line  
   (e) closed  
   (f) unbounded

19. (a) Domain: all points in the xy-plane  
   (b) Range: $z \geq 0$  
   (c) level curves: for $f(x,y) = 0$, the origin; for $f(x,y) = c > 0$, ellipses with center $(0,0)$ and major and minor axes along the x- and y-axes, respectively  
   (d) no boundary points  
   (e) both open and closed  
   (f) unbounded

20. (a) Domain: all points in the xy-plane  
   (b) Range: all real numbers  
   (c) level curves: for $f(x,y) = 0$, the union of the lines $y = \pm x$; for $f(x,y) = c \neq 0$, hyperbolas centered at $(0,0)$ with foci on the x-axis if $c > 0$ and on the y-axis if $c < 0$  
   (d) no boundary points  
   (e) both open and closed  
   (f) unbounded

21. (a) Domain: all points in the xy-plane  
   (b) Range: all real numbers  
   (c) level curves are hyperbolas with the x- and y-axes as asymptotes when $f(x,y) \neq 0$, and the x- and y-axes when $f(x,y) = 0$  
   (d) no boundary points  
   (e) both open and closed  
   (f) unbounded
22. (a) Domain: all \((x, y) \neq (0, y)\)
   (b) Range: all real numbers
   (c) level curves: for \(f(x, y) = 0\), the x-axis minus the origin; for \(f(x, y) = c \neq 0\), the parabolas \(y = cx^2\) minus the origin
   (d) boundary is the line \(x = 0\)
   (e) open
   (f) unbounded

23. (a) Domain: all \((x, y)\) satisfying \(x^2 + y^2 < 16\)
   (b) Range: \(z \geq \frac{1}{4}\)
   (c) level curves are circles centered at the origin with radii \(r < 4\)
   (d) boundary is the circle \(x^2 + y^2 = 16\)
   (e) open
   (f) bounded

24. (a) Domain: all \((x, y)\) satisfying \(x^2 + y^2 \leq 9\)
   (b) Range: \(0 \leq z \leq 3\)
   (c) level curves are circles centered at the origin with radii \(r \leq 3\)
   (d) boundary is the circle \(x^2 + y^2 = 9\)
   (e) closed
   (f) bounded

25. (a) Domain: \((x, y) \neq (0, 0)\)
   (b) Range: all real numbers
   (c) level curves are circles with center \((0, 0)\) and radii \(r > 0\)
   (d) boundary is the single point \((0, 0)\)
   (e) open
   (f) unbounded

26. (a) Domain: all points in the xy-plane
   (b) Range: \(0 < z \leq 1\)
   (c) level curves are the origin itself and the circles with center \((0, 0)\) and radii \(r > 0\)
   (d) no boundary points
   (e) both open and closed
   (f) unbounded
27. (a) Domain: all \((x, y)\) satisfying \(-1 \leq y - x \leq 1\)
(b) Range: \(-\frac{\pi}{2} \leq z \leq \frac{\pi}{2}\)
(c) level curves are straight lines of the form \(y - x = c\) where \(-1 \leq c \leq 1\)
(d) boundary is the two straight lines \(y = 1 + x\) and \(y = -1 + x\)
(e) closed
(f) unbounded

28. (a) Domain: all \((x, y)\), \(x \neq 0\)
(b) Range: \(-\frac{\pi}{2} < z < \frac{\pi}{2}\)
(c) level curves are the straight lines of the form \(y = cx\), \(c\) any real number and \(x \neq 0\)
(d) boundary is the line \(x = 0\)
(e) open
(f) unbounded

29. (a) Domain: all points \((x, y)\) outside the circle \(x^2 + y^2 = 1\)
(b) Range: all reals
(c) Circles centered at the origin with radii \(r > 1\)
(d) Boundary: the circle \(x^2 + y^2 = 1\)
(e) open
(f) unbounded

30. (a) Domain: all points \((x, y)\) inside the circle \(x^2 + y^2 = 9\)
(b) Range: \(z < \ln 9\)
(c) Circles centered at the origin with radii \(r < 9\)
(d) Boundary: the circle \(x^2 + y^2 = 9\)
(e) open
(f) bounded

31. f  
32. e  
33. a

34. c  
35. d  
36. b

37. (a)

(b)
46. (a)

\[ z = 1 - |x| - |y| \]

(b)

47. (a)

\[ z = \sqrt{x^2 + y^2 + 4} \]

(b)

48. (a)

\[ z = \sqrt{x^2 + y^2 - 4} \]

(b)

49. \( f(x, y) = 16 - x^2 - y^2 \) and \( \left( 2\sqrt{2}, \sqrt{2} \right) \Rightarrow z = 16 - \left( 2\sqrt{2} \right)^2 - \left( \sqrt{2} \right)^2 = 6 \Rightarrow 6 = 16 - x^2 - y^2 \Rightarrow x^2 + y^2 = 10 \)

50. \( f(x, y) = \sqrt{x^2 - 1} \) and \( (1, 0) \Rightarrow z = \sqrt{1^2 - 1} = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x = 1 \) or \( x = -1 \)

51. \( f(x, y) = \sqrt{x + y^2 - 3} \) and \( (3, -1) \Rightarrow z = \sqrt{3 + (-1)^2 - 3} = 1 \Rightarrow x + y^2 - 3 = 1 \Rightarrow x + y^2 = 4 \)

52. \( f(x, y) = \frac{2y - x}{x + y + 1} \) and \( (-1, 1) \Rightarrow z = \frac{2(1) - (-1)}{(-1) + 1 + 1} = 3 \Rightarrow 3 = \frac{2y - x}{x + y + 1} \Rightarrow y = -4x - 3 \)
53. \( f(x, y, z) = x^2 + y^2 + z^2 = 1 \)

54. \( f(x, y, z) = \ln \left( x^2 + y^2 + z^2 \right) = 0 \)

55. \( f(x, y, z) = x + z = 1 \)

56. \( f(x, y, z) = z = 1 \)

57. \( f(x, y, z) = x^2 + y^2 = 1 \)

58. \( f(x, y, z) = y^2 + z^2 = 1 \)

59. \( f(x, y, z) = z - x^2 - y^2 = 1 \)
   or \( z = x^2 + y^2 + 1 \)

60. \( f(x, y, z) = \frac{x^2}{25} + \frac{y^2}{16} + \frac{z^2}{9} = 1 \)
61. \( f(x, y, z) = \sqrt{x - y} - \ln z \) at \((3, -1, 1)\) \(\Rightarrow\) \(w = \sqrt{x - y} - \ln z\); at \((3, -1, 1)\) \(\Rightarrow\) \(w = \sqrt{3 - (-1)} - \ln 1 = 2\)
\[\Rightarrow \sqrt{x - y} - \ln z = 2\]

62. \( f(x, y, z) = \ln(x^2 + y + z^2) \) at \((-1, 2, 1)\) \(\Rightarrow\) \(w = \ln(x^2 + y + z^2)\); at \((-1, 2, 1)\) \(\Rightarrow\) \(w = \ln(1 + 2 + 1) = \ln 4\)
\[\Rightarrow \ln 4 = \ln(x^2 + y + z^2) \Rightarrow x^2 + y + z^2 = 4\]

63. \( g(x, y, z) = \sqrt{x^2 + y^2 + z^2} \) at \((1, -1, \sqrt{2})\) \(\Rightarrow\) \(w = \sqrt{x^2 + y^2 + z^2}\); at \((1, -1, \sqrt{2})\) \(\Rightarrow\) \(w = \sqrt{1^2 + (-1)^2 + (\sqrt{2})^2}\)
\[= 2 \Rightarrow 2 = \sqrt{x^2 + y^2 + z^2} \Rightarrow x^2 + y^2 + z^2 = 4\]

64. \( g(x, y, z) = \frac{x - y + z}{2x + y - z} \) at \((1, 0, -2)\) \(\Rightarrow\) \(w = \frac{x - y + z}{2x + y - z}\); at \((1, 0, -2)\) \(\Rightarrow\) \(w = \frac{1 - 0 + (-2)}{2(1) + 0 - (-2)} = -\frac{1}{4} \Rightarrow -\frac{1}{4} = \frac{x - y + z}{2x + y - z}\)
\[\Rightarrow 2x - y + z = 0\]

65. \( f(x, y) = \sum_{n=0}^{\infty} \left( \frac{x}{y} \right)^n = \frac{1}{1 - \left( \frac{x}{y} \right)} = \frac{y}{y - x} \) for
\[
\left| \frac{x}{y} \right| < 1 \Rightarrow \text{Domain: all points (x, y) satisfying } |x| < |y|;
\]
at \((1, 2)\) \(\Rightarrow\) since \(\left| \frac{x}{y} \right| < 1 \Rightarrow z = \frac{2}{2-1} = 2\)
\[\Rightarrow \frac{y}{y - x} = 2 \Rightarrow y = 2x\]

66. \( g(x, y, z) = \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!z^n} = e^{(x+y)/z} \Rightarrow \text{Domain: all points (x, y, z) satisfying } z \neq 0; \text{ at } (1, 0, 2)\)
\[\Rightarrow \frac{1}{1 - \left( \frac{x}{y} \right)} = \frac{x + y}{2x + y - z} \Rightarrow 6 = e^{(x+y)/z} \Rightarrow \frac{x+y}{z} = \ln 6\]

67. \( f(x, y) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - \theta^2}} = \sin^{-1}y - \sin^{-1}x \Rightarrow \text{Domain: all points (x, y) satisfying } -1 \leq x \leq 1 \text{ and } -1 \leq y \leq 1;\)
\(\text{at } (0, 1) \Rightarrow \sin^{-1}1 - \sin^{-1}0 = \frac{\pi}{2} \Rightarrow \sin^{-1}y - \sin^{-1}x\)
\[= \frac{\pi}{2}. \text{ Since } -\frac{\pi}{2} \leq \sin^{-1}y \leq \frac{\pi}{2} \text{ and } -\frac{\pi}{2} \leq \sin^{-1}x \leq \frac{\pi}{2}, \text{ in order for } \sin^{-1}y - \sin^{-1}x \text{ to equal } \frac{\pi}{2}, 0 \leq \sin^{-1}y \leq \frac{\pi}{2} \text{ and } -\frac{\pi}{2} \leq \sin^{-1}x \leq 0; \text{ that is } 0 \leq y \leq 1 \text{ and } -1 \leq x \leq 0. \text{ Thus}\)
\[y = \sin\left( \frac{\pi}{2} + \sin^{-1}x \right) = \sqrt{1 - x^2}, x \leq 0\]

68. \( g(x, y, z) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin(\theta)}{\sqrt{1 - \sin^2(\theta)}} = \tan^{-1}y - \tan^{-1}x + \sin^{-1}\left( \frac{\sqrt{2}}{2} \right) \Rightarrow \text{Domain: all points (x, y, z) satisfying } -2 \leq z \leq 2;\)
\(\text{at } \left( 0, 1, \sqrt{3} \right) \Rightarrow \tan^{-1}1 - \tan^{-1}0 + \sin^{-1}\left( \frac{\sqrt{3}}{2} \right) = \frac{\pi}{2} \Rightarrow \tan^{-1}y - \tan^{-1}x + \sin^{-1}\left( \frac{\sqrt{3}}{2} \right) = \frac{\pi}{2}. \text{ Since } -\frac{\pi}{2} \leq \sin^{-1}\left( \frac{\sqrt{3}}{2} \right) \leq \frac{\pi}{2},\)
\[\frac{\pi}{12} \leq \tan^{-1}y - \tan^{-1}x \leq \frac{\pi}{12} \Rightarrow z = 2 \sin\left( \frac{\pi}{12} - \tan^{-1}y + \tan^{-1}x \right), \frac{\pi}{12} \leq \tan^{-1}y - \tan^{-1}x \leq \frac{\pi}{12}\]

69-72. Example CAS commands:

Maple:

```maple
with(plots);
f := (x,y) -> x*sin(y/2) + y*sin(2*x);
xdomain := x=0..5*Pi;
ydomain := y=0..5*Pi;
x0,y0 := 3*Pi,3*Pi;
plot3d(f(x,y), xdomain, ydomain, axes=boxed, style=patch, shading=zhue, title="#69(a) (Section 14.1)" );
```